Homework 5

This assignment is due by **11:59pm on Friday 10/11** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, save your solutions as a single pdf and upload it to gradescope. Be sure to **assign your pages** on the gradescope upload page; failure to do so will result in a penalty.

You may collaborate with your peers on homework assignments and to discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit (with the exception of row operations).

1. Use the $(7, 4)$ Hamming code defined in class with

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

to answer the following questions.

(a) Suppose Alice wishes to encode the message $M = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. What is C_A?

(b) Suppose Bob receives the messages below. For each message, determine the original message M.

$$(i)\ C_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad (ii)\ C_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

2. In this problem, we will show that any message sent using the $(7,4)$ Hamming code can be uniquely decoded. Define the *Hamming distance* between two vectors to be the number of components in which they differ. For example, the Hamming distance between $\begin{bmatrix}0\\1\\0\\0\\1\\1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\\0\\1\\1\\0\\0\end{bmatrix}$ is three (since components 1, 3, and 7 are different). A *codeword* is a vector of the form $C_A = GM$ for some original message M (these are the vectors generated by the Hamming code; i.e. vectors in the column space of G).

   (a) Show that two codewords in the $(7,4)$ Hamming code cannot be at Hamming distance 1 from each other. Hint: Use the following facts: if C_1 and C_2 are two codewords at distance 1 then for some i, $C_1 = C_2 + \mathbf{e}_i \pmod 2$; and codewords must be in the null space of H.

   (b) Show that two codewords in the (7,4) Hamming code cannot be at Hamming distance 2 from each other. Hint: Use an argument similar to part (a).

   (c) The above parts show that the Hamming distance between any two codewords must be at least 3. Explain why this implies that any message can be uniquely decoded.

3. (a) Let V and W be subspaces of $\mathbb{R}^n$. Show that if $V \perp W$, then $V \cap W = \{\mathbf{0}\}$.

   (b) Use part (a) to deduce that any two subspaces of $\mathbb{R}^3$ represented by planes cannot be orthogonal.

4. Let $V = \left\{ \begin{bmatrix}2x+y\\x-y\\3x\\y\end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$.

   (a) Interpret V as the column space of a matrix A and use this to find a basis for V.

   (b) Use part (a) to determine $V^\perp$ and find a basis for $V^\perp$.

   (c) Verify your answer to part (b) is correct by taking all possible dot products of the basis vectors of V with the basis vectors of $V^\perp$. [Note that orthogonality of the bases implies orthogonality of the entire spaces].

   (d) What do V and $V^\perp$ look like geometrically? How does this compare to Problem 3b?

5. The set of solutions to the linear equation $2x + y - 3z = 0$ is a plane W in $\mathbb{R}^3$ passing through the origin.

   (a) Interpret W as the nullspace of a matrix A. What is A?

(b) Find a basis for $W = N(A)$. To make your calculations easier in the next parts, choose a basis that does not contain fraction.

(c) Find the projection of $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ onto W using the basis you found in part (b).

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(\mathbf{v}) = \text{proj}_W \mathbf{v}$. Find the projection matrix P associated to T.

(e) Determine the rank of P without any row reduction; explain your reasoning.

6. Let A be an $m \times n$ matrix, and let $P = A(A^TA)^{-1}A^T$ be the projection matrix associated to the linear transformation that projects vectors in $\mathbb{R}^m$ onto the column space of A.

(a) Prove that $P^2 = P$

(b) Interpret part (a) geometrically, in terms of projections. That is, explain why $P(P\mathbf{b})$ always equals $P\mathbf{b}$.

(c) What is wrong with the following "proof" that $P = I$? (Note that this statement is generally false!)

Proof:

$$\begin{aligned} P &= A(A^TA)^{-1}A^T \\ &= A((A^{-1}(A^T)^{-1})A^T \\ &= (AA^{-1})((A^T)^{-1}A^T) \\ &= (I)(I) \\ &= I \end{aligned}$$

□

7. Find the line of best fit through the data points $(1, 1)$, $(2, 1)$, and $(3, 0)$ using the method discussed in class.